#4 Darts - Formulation

Author: Andrea Rujano
August, 2021

This is the MIP formulation of the puzzle. Statement and solution implementation of all puzzles are available from the main page of the Fun Puzzles project, which is maintained by Mip Master.

Let's start by reviewing the statement of the puzzle:
There are three people darts: Andrea, Antonio, and Luiz. They threw 6 darts each (red marks in the figure), and each scored 71 points. We also know that Andrea's first 2 darts scored 22 points and Antonio's first dart scored 3 points. We need to find out which player hit the bullseye.

Darts

Input Data

We start by defining three sets of indices.

- The first set corresponds to the **player** (Andrea, Antonio, and Luiz, respectively):
  `I = {1, 2, 3}`
- Next, we define the set of **shots** (each player threw 6 darts):
  `J = {1, 2, 3, 4, 5, 6}`
- Next, we define the set of **scores** (scoring regions of the dartboard):
  `K = {1, 2, 3, 5, 10, 20, 25, 50}`
- Finally, we define a dictionary which indicates the numbers of **marks** in each **scoring region** (as shown in the figure):
  `R = {1: 3, 2: 2, 3: 2, 5: 2, 10: 3, 20: 3, 25: 2, 50: 1}`

Decision variables

With the sets of indices defined above we can now define the decision variables. Can you think how we do this?

We want to answer is this: What is the score of each player in each shot?

So we define decision variables with three indices, and they are all binary variables:

- x_{ijk} equals 1 if player i scores k in shot j, 0 otherwise.

For example, if the variable $x_{1,2,5}$ equals one, it means that player 1 scores 5 in his second shot. We assume that in every shot the player hits the dartboard, so she always scores something.

Constraints

With the definition of the decision variables above, we can now model the rules of the game with constraints.

- *Every shot hits one, and only one, scoring region*.

$$\sum_k x_{ijk} = 1, \ \forall i, j.$$

 For example, for $i = 1$ and $j = 2$, this constraint becomes

$$x_{1,2,1} + x_{1,2,2} + x_{1,2,3} + x_{1,2,5} + x_{1,2,10} + x_{1,2,20} + x_{1,2,50} = 1,$$

 which means that, out of the seven possible scores, the first player in her second shot can only hit one score. In fact, the only why for a sum of binary variables to add up to one is that one of those variables equal one and all the others equal zero.

- *Each player scored 71 points*.

$$\sum_{j,k} k \cdot x_{ijk} = 71, \ \forall i.$$

 This constraint is saying that the total score of player i, across all shots and all possible scores in each shot, must add up to 71. Notice that we multiply x_{ijk} by k so that, if $x_{ijk} = 1$, we add k to the sum, and if $x_{ijk} = 0$, we add zero.

- *Number of marks in each scoring region*.

$$\sum_{i,j} x_{ijk} = R[k], \ \forall k.$$

 This is saying that for $k = 25$, for example, there are $R[25] = 2$ darts, across all player and all shots, that hit the scoring region of 25 points. This constraint, in combination with the next two constraints, makes the solution to this puzzle to be unique.

- *Andrea's first two shots scored 22 points*.

$$\sum_k x_{11k} + x_{12k} = 22.$$

 This constraint is identical to constraint "*Each player scored 71 points*", except that this one is restricted to player 1 and its first two shots.
 Alternatively, given that 20 points can only be achieved in two shots if one dart hits 20 and the other dart hits 2, we could simply enforce $x_{1,1,20} = 1$ and $x_{1,2,2} = 1$ (or $x_{1,1,2} = 1$ and $x_{1,2,20} = 1$).

- *Antonio's first shot scored 3 points*.

$$x_{2,1,3} = 1.$$

Objective Function

This is just a feasibility problem, since there is nothing to optimize. Hence, we can simply define a dummy objective function such as the sum of all variables:

$$\min \sum_{i,j,k} x_{ijk}.$$

Final formulation

$$\begin{aligned}
\min \quad & \textstyle\sum_{i,j,k} x_{ijk} \\
\text{s.t.} \quad & \textstyle\sum_k x_{ijk} = 1, \quad \forall i, j \\
& \textstyle\sum_{j,k} k \cdot x_{ijk} = 71, \quad \forall i \\
& \textstyle\sum_{i,j} x_{ijk} = R[k], \quad \forall k \\
& \textstyle\sum_k x_{11k} + x_{12k} = 22 \\
& x_{2,1,3} = 1 \\
& x_{ijk} \in \{0, 1\} \quad \forall i, j, k.
\end{aligned}$$