#3 Digits Tracking - Formulation

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July, 2021

This is the MIP formulation of the puzzle. Statement and solution implementation of all puzzles are available from the main page of the Fun Puzzles project, which is maintained by Mip Master.

The goal is to fill digits into a grid of ten numbered cells in a way that the digit in each cell i represents the number of times that the digit i appears on the grid.

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
?	?	?	?	?	?	?	?	?	?

If we put the digit 3 in the cell number 5, for instance, it means that we have to find a way to put digit 5 in 3 different cells of our grid. Got it? Nice. So let's think about how can we ask the solver to find this digits for us, shall we?

Input Data

We start by defining two set of indices (which turn out to be the same but we keep them separate for clarity):

- Set of columns:
  `I = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}`
- Set of digits:
  `K = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}`

Decision Variables

There is one decision to be made: What digit goes in each cell?

So we define one set of binary variables:

- x_{ik} equals 1 if digit k goes in cell i, 0 otherwise.

Constraints

Two constraints are required to model this puzzle.

- *Exactly one digit must be assigned to each cell*:

$$\sum_{k \in K} x_{ik} = 1, \qquad \forall i.$$

  These constraints say that, given a cell i, if we sum the x variables assigned across the digits, the result has to be 1, i.e., only one of the variables x assigned to cell i can take value 1.

If you are not familiar with summation notation, it may be helpful to look at an example. For instance, let's look at $i = 2$. In this case we have $x_{20} + x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} + x_{28} + x_{29} = 1$, which means that one of the variables in this expression takes value 1 and all the others take value 0.

- *If you assign the digit k for the cell i, then the digit i must appear k times in the grid*:

$$\sum_{k \in K} k \cdot x_{ik} = \sum_{k \in K} x_{ki}, \qquad \forall i.$$

These are the constraints that capture the soul of the puzzle. We are saying here that for a given cell i, the digit k that we assign to that cell, is also the number of times that digit i must appear in the entire grid. Let's have a closer look into this constraint. How about we pick a cell to better understand this?
All right! For example, if we look to cell number 2, i.e., $i = 2$, then the expression on the left hand side would become

$$0 \cdot x_{20} + 1 \cdot x_{21} + 2 \cdot x_{22} + 3 \cdot x_{23} + 4 \cdot x_{24} + 5 \cdot x_{25} + 6 \cdot x_{26} + 7 \cdot x_{27} + 8 \cdot x_{28} + 9 \cdot x_{29}.$$

But we know that only one of this variables would be 1 while all the other would be 0, right? So, the result of this expression would be exactly the digit that we assign for cell number 2. Amazing, right?
Now, let's look closely to the expression on the right side. Did you realized that we wrote x_{ki} instead of x_{ik}? Cool. We did that so we can count how many times the digit 2, in our example, appears in the entire grid. In this case, the right hand side expression then becomes

$$x_{02} + x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} + x_{72} + x_{82} + x_{92}.$$

Each of this variables would be 1 only if the digit 2 was assigned to that specific cell. So, the result of this sum would be the number of times that digit 2 appears in the entire grid. That is exactly what we need, because we want that the digit assigned to cell 2 is also the number of times that digit 2 appears in the grid.

Objective Function

There is no objective to maximize or minimize in this problem. We only need to find one feasible solution (which turns out to be unique in this case). But there is no problem if we define an objective function like this:

$$\min x_{11}.$$

Final Formulation

$$\begin{aligned} \min \quad & x_{11} \\ \text{s.t.} \quad & \sum_{k \in K} x_{ik} = 1, \quad \forall i, \\ & \sum_{k \in K} k \cdot x_{ik} = \sum_{k} x_{ki}, \quad \forall i, \\ & x_{ik} \quad \in \{0,1\}, \forall i, k. \end{aligned} \tag{1}$$